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Extension of Gauss' Method for the Solution of Kepler's Equation

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Gauss' method for solving Kepler's equation is extended to arbitrary epochs and orbital eccentricities. Although originally developed for near parabolic orbits in the vicinity of pericenter, a generalization of the method leads to a highly efficient algorithm which compares favorably to other methods in current use. A key virtue of the technique is that convergence is obtained by a method of successive substitutions with an initial approximation that is independent of the orbital parameters. The equations of the algorithm are universal, i.e., independent of the nature of the orbit whether elliptic, hyperbolic, parabolic or rectilinear.

Introduction

In his *Theoria Motus,* Gauss[1] reported the development of an extremely efficient technique for solving Kepler's equation in the case of near parabolic orbits with the time interval measured from the instant of pericenter passage. Gauss was apparently quite impressed with the ingenuity of his method—so much so that he laboriously prepared extensive tables, which are required to implement the algorithm, and included them in an appendix of his book.

The essence of Gauss' idea is a clever transformation of the Kepler equation to a form resembling a third-order algebraic expression. The transformed equation is exactly a cubic for parabolic motion and nearly so for elliptic and hyperbolic orbits whose eccentricities are close to unity. The solution is obtained by successive approximations. At each stage 1) the equation is solved as though it were a cubic and 2) the tables are consulted to revise an algebraic coefficient. Convergence is remarkably rapid, with typically two iteration steps sufficing for seven decimal places of accuracy.

The purpose of this paper is to explore the possibility of extending Gauss' method to the general case for which the time interval is not reckoned from pericenter and the orbital eccentricity is arbitrary. Gauss' auxiliary tables can be replaced by economized power series expansions whose coefficients are exactly obtainable by a symbol manipulating computer program. The resulting algorithm is necessarily more complex than the original because of both the arbitrary epoch and the annoying fact that the more general quasicubic equation can have multiple real roots which must be reconciled.

Nevertheless, the overall program logic is simple and convergence is relatively fast when compared with the more standard methods in use today. Indeed, when the proposed method is contrasted with a subroutine having the same capabilities and originally a part of the software for the avionics computers of the NASA Space Shuttle orbiter vehicle, the results are generally encouraging in both memory utilization and execution time.

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Transformation of Kepler's Equation

Let r_0, v_0 be the position and velocity vectors corresponding to an epoch time t_0 for a Keplerian orbit whose gravitational constant is μ and define

$$T=\sqrt{\mu}(t-t_0) \qquad \sigma_0=r_0\cdot v_0/\sqrt{\mu} \tag{1}$$

where t represents an arbitrary time in the orbital motion. Then Kepler's equation for both elliptic and hyperbolic orbits may be written as

$$T=\sqrt{\pm a}[\pm(a-\alpha r_0)P+r_0Q]\pm a\sigma_0R \tag{2}$$

where

$$P=\begin{cases}(E-E_0)-\sin(E-E_0)\\ \sinh(H-H_0)-(H-H_0)\end{cases} \tag{3}$$

$$Q=\begin{cases}\alpha(E-E_0)+(1-\alpha)\sin(E-E_0)\\ \alpha(H-H_0)+(1-\alpha)\sinh(H-H_0)\end{cases} \tag{4}$$

$$R=\begin{cases}1-\cos(E-E_0)\\ \cosh(H-H_0)-1\end{cases} \tag{5}$$

and a,E,H denote, respectively, the semimajor axis, the eccentric anomaly for the ellipse, and the analogous quantity for the hyperbola. The constant α is, for the moment, unspecified – we simply note in passing that $\alpha=0$ yields the standard form[2] of Kepler's equation referenced to an arbitrary epoch. (In Eq. (2), as well as in the rest of this section and the next, the choice of upper or lower sign is made according as the orbit is, respectively, an ellipse or hyperbola.)

With the motivation provided by Gauss, the next step in the transformation is to define

$$\gamma_0=r_0/a=2-r_0v_0^2/\mu \tag{6}$$

$$\beta_0=\tfrac{1}{2}(1-\alpha\gamma_0) \tag{7}$$

$$y=\pm 6P/Q=\gamma_0w^2 \tag{8}$$

$$B=(Q^3/6P)^{\frac{1}{2}} \tag{9}$$

$$D=2R/(6PQ)^{\frac{1}{2}} \tag{10}$$

so that Eq. (2) becomes

$$T=r_0^{3/2}B[\tfrac{1}{3}\beta_0w^3+\tfrac{1}{2}(\sigma_0/r_0^{\frac{1}{2}})Dw^2+w] \tag{11}$$

This is the generalization of the form first obtained by Gauss. As he considered only the elementary version of Kepler's equation, the pericenter distance was r_0 and the w^2 term was missing ($\sigma_0=0$). Hence, there was no need to introduce the D function.

The quantities B and D defined by Eqs. (9) and (10), may be expressed as power series expansions

$$B=1\pm(1/4)(\alpha-9/10)z^2+\ldots \tag{12}$$

$$D=1\pm(1/12)(3/10-\alpha)z^2+\ldots \tag{13}$$

where z is identified with either $E-E_0$ or $H-H_0$ according to the nature of the orbit. In his development, Gauss chose α to be 9/10 so that the factor B in Eq. (11) would be as insensitive as possible to changes in the anomaly. If we, too, make this choice, then

$$B=1 \qquad +\frac{3}{2800}z^4\mp\frac{1}{84{,}000}z^6\pm\ldots$$

$$D=1\mp\frac{1}{20}z^2+\frac{1}{4200}z^4\pm\frac{11}{504{,}000}z^6\mp\ldots$$

The quantity y, defined in Eq. (8), may also be expanded as a power series. We have, for $\alpha=9/10$,

$$y=\pm z^2\left(1\mp\frac{1}{30}z^2-\frac{1}{5040}z^4\pm\frac{1}{36{,}000}z^6+\ldots\right)$$

Therefore, by the process of series reversion, we may obtain B and D as the following power series representations in y:

$$B=1 \qquad +\frac{3}{2800}y^2+\frac{1}{16{,}800}y^3+\ldots \tag{14}$$

$$D=1-\frac{1}{20}y-\frac{1}{700}y^2-\frac{1}{12{,}000}y^3-\ldots \tag{15}$$

Equations (11, 14, and 15) are the essence of the extended method of Gauss. From an initial approximation for y, values of B and D are calculated from the series expansions. Equation (11) is then solved as an algebraic cubic for w resulting in a new, improved value for y determined from $y=\gamma_0 w^2$ [Eq. (8)].

It is important to remark that the fundamental relations involved are universal, i.e., one set of equations is valid for all orbits including even the rectilinear. The sign of γ_0 determines the sign of y: positive for the ellipse and negative for the hyperbola. The parabola corresponds to the case $y=0$. It is easy to demonstrate for this value that Eq. (11) is, indeed, the generalized form of Barker's equation with

$$w=\sqrt{p/r_0}\,(\tan\tfrac{1}{2}f-\tan\tfrac{1}{2}f_0)$$

where p is the orbital parameter and f is the true anomaly.

Series Representations

Gauss' method of successive approximations to the solution of the elementary form of Kepler's equation depended, for its efficiency, on the relative insensitivity of function B to changes in the anomaly. His choice of $\alpha=9/10$ insured that B would differ from unity by a quantity of fourth order in the anomaly. In the more general case, with two functions of the anomaly B and D to contend with, the choice of α is not so straightforward.

Referring to Eqs. (12) and (13) we observe several possibilities: 1) choose $\alpha=9/10$ to render B as nearly constant as possible; 2) choose $\alpha=3/10$ so that D will have this characteristic; or 3) choose $\alpha=3/4$ so that the behavior of B and D will be identical for second order variations in the anomaly. The algorithm was exercised[3] for each of these cases. It was substantiated, thereby, that the value originally assigned by Gauss to enhance convergence is also the proper choice for the general problem.

It is not surprising that the most time-consuming and tedious aspects of the algorithm are the two power series evaluations of Eqs. (14) and (15). (Gauss simplified this task by preparing a set of tables suitable for manual computation.) Fortunately, it is possible to determine both B and D from a *single* series expansion as we now demonstrate.

For this purpose, Eqs. (3, 4, 8, and 9), with $\alpha=9/10$, are first used to establish

$$\left.\begin{matrix}\sin z\\ \sinh z\end{matrix}\right\}=Q\mp\frac{9}{10}P=Q\left(1-\frac{3}{20}y\right)$$
$$=B\sqrt{\pm y}\left(1-\frac{3}{20}y\right)$$

With this result, together with the definition of D in Eq. (10), we have

$$D=\frac{2}{yB}\begin{cases}\sin z\tan\tfrac{1}{2}z\\ -\sinh z\tanh\tfrac{1}{2}z\end{cases}$$

or

$$D=\frac{1}{K}\left(1-\frac{3}{20}y\right) \tag{16}$$

where the function K is defined by

$$K=\begin{cases}\sqrt{y}/2\ \tan\tfrac{1}{2}z\\ \sqrt{-y}/2\ \tanh\tfrac{1}{2}z\end{cases} \tag{17}$$

Next, from Eqs. (8-10), we find that

$$BD=\frac{2R}{\pm y}=\frac{1}{K^2+\tfrac{1}{4}y} \tag{18}$$

Hence, both B and D can be simply calculated in terms of the quantity K. The final task here is to obtain the expansion of K as a power series in y.

Consider first the elliptic case and use Eq. (3) with $z=E-E_0$ to write

$$P=\sin z\sec^2\tfrac{1}{2}z\left(\frac{\tfrac{1}{2}z}{\tan\tfrac{1}{2}z}-\frac{1}{1+\tan^2\tfrac{1}{2}z}\right)$$

For brevity, τ is used to denote $\tan^2\tfrac{1}{2}z$ and the terms in parenthesis are expanded as a power series in τ. Thus

$$\left(1-\frac{1}{3}\tau+\frac{1}{5}\tau^2-\ldots\right)-\left(1-\tau+\tau^2-\ldots\right)$$
$$=\frac{2}{3}\left(\tau-\frac{6}{5}\tau^2+\frac{9}{7}\tau^3-\frac{12}{9}\tau^4+\ldots\right)$$

Similarly, for Q from Eq. (4), we have

$$Q=\sin z\sec^2\tfrac{1}{2}z\left(1-\frac{6}{15}\tau+\frac{7}{25}\tau^2-\frac{8}{35}\tau^3+\ldots\right)$$

Then, since $y=6P/Q$, we have y represented as the ratio of two power series in τ. By reversing the series, we obtain

$$K^2=\frac{y}{4\tau}=1-\frac{1}{5}y+\frac{1}{350}y^2+\frac{1}{4200}y^3+\ldots$$

The square root of the series for K^2 produces finally

$$K=1-\frac{1}{10}y-\frac{1}{280}y^2-\frac{1}{4200}y^3-\ldots \tag{19}$$

As would be anticipated, the same power series for K results when one addresses explicitly the hyperbolic case. The details are, therefore, omitted.

It is of interest to note that Gauss was also concerned with the series for K^2, obtaining the expansion by hand to terms of order y^5 and tabulating the values. However, K^2 was not used for the iterative portion of his algorithm but only to produce finally the radius vector after convergence. In retrospect, it appears that Gauss undoubtedly knew of the relationship by which B could be computed from K. He gave explicitly only two terms in the expansion of B and six in the expansion of K^2; yet functions of both to the same accuracy are to be found in his tables.‡

In Table 1 of this paper the first 23 coefficients of the power series expansion of K are listed which were obtained by a symbol manipulating computer program.[4] This number of terms is required for 16 decimal places of accuracy if y is permitted the range $-2 \leq y \leq 2$. Fewer are needed for a narrower range and/or less precision. This topic is discussed more fully in a later section.

Solution of the Cubic Equation

The solution of the transformed elementary version of Kepler's equation considered by Gauss presents no problem except for rectilinear orbits with which he was not concerned. In his case, with r_0 corresponding to pericenter, the factor β_0, appearing as the coefficient of w^3 in Eq. (11), is easily shown to be

$$\beta_0 = (1+9e)/20$$

where e is the orbital eccentricity. Clearly, β_0 is positive nonzero for all orbits. Furthermore, the resulting cubic equation for w has one and only one positive real root.

In the more general case, however, two difficulties arise which must be addressed. First, we observe from Eqs. (6) and (7) that β_0 will vanish at any point in an orbit for which

$$r_0 v_0^2/\mu = 8/9$$

Second, due to the presence of σ_0 in the coefficient of w^2, which may be either positive or negative, the cubic equation can possess three real roots.

The first of these problems may be successfully countered by a simple change of variable which, at the same time, will also convert the cubic equation to its normal form. To this end, we substitute x for w in Eq. (11), where

$$w = \frac{3T}{r_0^{3/2} B(1+x/d)} \tag{20}$$

and d is an arbitrary constant at our disposal. The resulting cubic equation for x is simply

$$x^3 - 3\epsilon x - 2b = 0 \tag{21}$$

where

$$\epsilon = d^2\left(1 + \frac{3\sigma_0 DT}{2r_0^2 B}\right) \tag{22}$$

$$2b = d^3\left[2 + \frac{9T}{r_0^2 B}\left(\frac{\sigma_0 D}{2} + \frac{\beta_0 T}{r_0 B}\right)\right] \tag{23}$$

By choosing d to be either plus or minus one, we can assure that b will never be negative. Then, although ϵ can still have either sign, Eq. (21) will have exactly one non-negative real root.

‡Although the definition of B in this paper is the same as that used by Gauss, there are some notational differences to be reconciled. Gauss' tables give log B and C as functions of A over the interval $0 \leq A \leq 0.3$ for the ellipse and hyperbola, separately. The relation to our notation is $\pm y = 4A$ and $K^2 = 1 - y/5 + C$.

Table 1 Coefficients of power series in y for $K = 1 - \sum_{n=1}^{\infty} k_n (y/10)^n$

n	k_n
1	1
2	$\frac{5}{14}$
3	$\frac{5}{21}$
4	$\frac{2483}{12936}$
5	$\frac{4769}{28028}$
6	$\frac{564131}{3531528}$
7	$\frac{4159187}{26682656}$
8	$\frac{10466541797}{66920101248}$
9	$\frac{24133416739}{150570227808}$
10	$\frac{210510810747199}{1260573947208576}$
11	$\frac{6223890626368873}{35296070521840128}$
12	$\frac{5016113766441671}{26645072844918528}$
13	$\frac{21317495853523495939}{105088167300358674432}$
14	$\frac{569387294869988659818 29}{25844683278068209998643 2}$
15	$\frac{1311210690427691562223414 67}{5442890298361165025714257 92}$
16	$\frac{2557661479664255756608281 3697}{9652058795760465978933284 0448}$
17	$\frac{57550946139904443174967879 02529}{19641939649372548267129233 031168}$
18	$\frac{21310081228415003781667708 68319}{65473132164575160890430776 77056}$
19	$\frac{70169749142311812573613776 27542689}{19327668614982587494855165 302669312}$
20	$\frac{38887674239869347644813839 157191658164 25}{95670896622389984057220851 214121447587 84}$
21	$\frac{14730136134243221759065712 6971029306740 75}{32259277538789349996153807 6821169839800 32}$
22	$\frac{17162172862067846092625659 078536724251642 499}{33356092975108187896023037 143308961435353 088}$

To show that this non-negative real root is the proper one under all circumstances, we first observe that, for $T=0$, Eq. (21) has three real roots—one at $x=2$ and a double root at $x=-1$. Only the positive root is appropriate for calculating w from Eq. (20). Now, by allowing T to increase or decrease, a simple continuity argument will suffice to prove that the non-negative real root of Eq. (21) is always the correct choice.

Finally, the possibility of a singularity arising in Eq. (20) must be considered, i.e., could $x=1$ be a root of Eq. (21) when $d=-1$? Since w must have a finite value for all non-parabolic orbits, we need be concerned only that this might occur for the parabola. However, from Eqs. (21-23), it is easy to show that $x=1$ and $d=-1$ requires $\beta_0=0$ for nonzero values of T. But $\beta_0=\frac{1}{2}$ for the parabola so that the potential singularity is nonexistent.

The classical explicit formulas for obtaining the roots of a cubic can, of course, be used for solving Eq. (21). They are somewhat cumbersome in that different formulas are required depending on the number of real roots. The authors have found that a simple Newton iteration is more suitable for an efficient computer mechanization, especially since the accuracy with which x must be obtained is considerably less than that required for y. With an initial value for x selected as

$$x_0=1+|\epsilon|$$

we can insure that convergence of the Newton algorithm to the appropriate root is inevitable.

Algorithm for the Kepler Problem

The complete algorithm for the solution of Kepler's problem is elegant in its simplicity. The relevant equations are summarized in this section in a form designed to minimize the arithmetic operations required in a computer mechanization. Certain auxiliary quantities are introduced for this purpose solely to effect a more efficient and compact algorithm.

Given the state vectors $\boldsymbol{r}_0$, $\boldsymbol{v}_0$ at time t_0, we desire the corresponding state $\boldsymbol{r}$, $\boldsymbol{v}$ at some other time t. The gravitation constant μ may be effectively eliminated from the equations by first defining

$$\boldsymbol{\nu}_0=\boldsymbol{v}_0/\sqrt{\mu} \qquad T=\sqrt{\mu}(t-t_0) \tag{24}$$

We then begin the solution by calculating the preliminary quantities

$$\sigma_0=\boldsymbol{r}_0\cdot\boldsymbol{\nu}_0 \qquad \gamma_0=2-r_0\nu_0^2 \tag{25}$$

$$\beta_0=\tfrac{1}{2}-9\gamma_0/20 \tag{26}$$

The iteration to determine y starts with the initial value $y=0$ corresponding to a parabolic orbit. Hence, the initial values of B and D are both unity. (It will be more convenient here to use the quantity δ instead of B where $\delta=1/BD$.)

With $D=\delta=1$, we next calculate

$$\xi=3\delta DT/r_0 \qquad \eta=\xi/r_0 \qquad \zeta=\tfrac{1}{2}\sigma_0 D \tag{27}$$

from which the coefficients in the cubic equation for x are obtained from

$$\epsilon=1+\eta\zeta \qquad b=|\epsilon+\tfrac{1}{2}\eta(\zeta+\beta_0\xi)| \tag{28}$$

A sequence of approximations $x_0, x_1, x_2, \ldots$ to the appropriate root is generated recursively using a Newton iteration

$$x_0=1+|\epsilon| \qquad x_{n+1}=\frac{2}{3}\,\frac{x_n^3+b}{x_n^2-\epsilon} \tag{29}$$

After convergence, a new value of y is computed from

$$\theta=\xi/(1\pm x) \qquad \phi=\theta^2/r_0 \qquad y=\gamma_0\phi \tag{30}$$

(The choice of sign in the equation for θ depends on the sign of the quantity within the absolute value symbols in the equation for b, i.e., the plus sign is chosen if that quantity is non-negative and the minus sign otherwise.)

Using the new value of y, the power series for $K(y)$ is evaluated using the coefficients listed in Table 1. We then calculate new values for D and δ from

$$D=(1/K)(1-3y/20) \qquad \delta=K^2+\tfrac{1}{4}y \tag{31}$$

and repeat the computation, beginning with Eqs. (27), until y ceases to change by a preassigned amount. (During the second and subsequent cycles through these equations it is, of course, more efficient to select for x_0 the last value of x determined from the previous Newton iteration.)

The elements of the state transition matrix $\Phi(t,t_0)$, needed to extrapolate $\boldsymbol{r}$, $\boldsymbol{\nu}$ from $\boldsymbol{r}_0$, $\boldsymbol{\nu}_0$, are most conveniently obtained in terms of the auxiliary quantities

$$\lambda=\tfrac{1}{2}\phi/\delta \qquad \chi=1-\gamma_0\lambda$$
$$\psi=r_0\lambda \qquad \omega=\theta K/\delta \tag{32}$$

(We do not include the details of the derivation, since the reader should have no difficulty verifying the relations from the standard two-body equations.[2]) The magnitude of the new position vector $\boldsymbol{r}$ is then

$$r=r_0\chi+\psi+\sigma_0\omega \tag{33}$$

and the transition matrix

$$\Phi(t,t_0)=\begin{bmatrix} 1-\lambda & r_0\omega+\sigma_0\psi \\ -\omega/r_0 r & 1-\psi/r \end{bmatrix} \tag{34}$$

The description of the basic algorithm is now complete. We note that no square or cube roots and only one polynomial are involved in the calculations. However, we must recall that the efficiency and practicality of the method require that values of y (hence, also, values of $t-t_0$) be kept within reasonable bounds.

In order to deal with the general problem in which the time interval is unrestricted, let us define y_m (and, correspondingly, T_m) as the maximum permissible values of y and T. Then, the algorithm is easily modified, as will now be described.

Each time a new value of y is computed in Eqs. (30), a test is made to determine if $|y|\le y_m$. If the test fails, we set the magnitude of y to the value of y_m, leaving the sign unchanged, and then compute

$$\phi_m=y/\gamma_0 \qquad \theta_m=\sqrt{\phi_m r_0}$$
$$T_m=[(\tfrac{1}{3}\beta_0\theta_m+\tfrac{1}{2}\sigma_0 D_m)\theta_m+r_0]\theta_m/\delta_m D_m \tag{35}$$

followed by Eqs. (32-34). (Division by γ_0 does not present a problem. If γ_0 were zero, as it is for parabolic orbits, the test in question would not have failed.) Corresponding values of σ_m and γ_m are obtained from

$$\sigma_m=\sigma_0\chi_m+(1-\gamma_0)\omega_m \qquad \gamma_m=r_m\gamma_0/r_0 \tag{36}$$

Then, by replacing T by $T-T_m$ and r_0, σ_0, γ_0 by r_m, σ_m, γ_m, we are prepared to restart the algorithm anew, beginning with Eq. (26).

If the time difference $t-t_0$ is sufficiently large, this process of decrementing T may have to be repeated several times. The transition matrices thus sequentially generated are, of course, multiplied together to produce the final desired matrix.

Results

The algorithm described in the previous section was exercised for a variety of representative orbits. In each case,

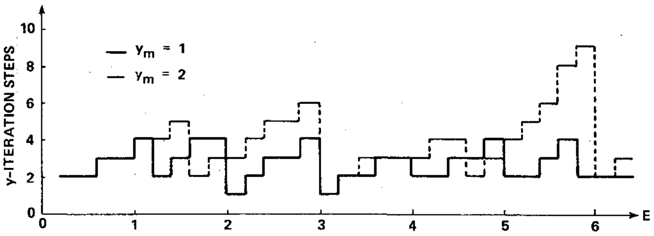

Fig. 1 Number of steps in y-iteration vs E for elliptic orbit ($e = 0.5$).

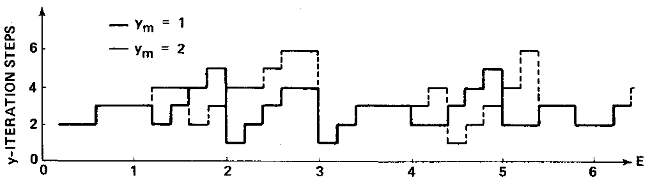

Fig. 2 Number of steps in y-iteration vs E for elliptic orbit ($e = 0.8$).

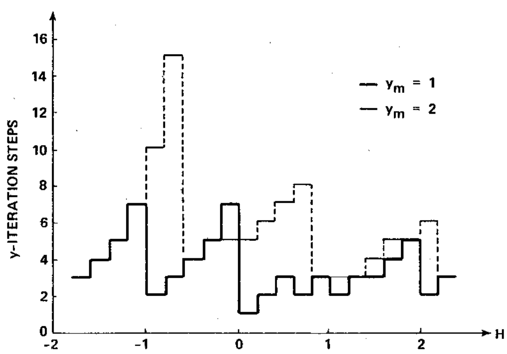

Fig. 3 Number of steps in y-iteration vs H for hyperbolic orbit ($e = 2$).

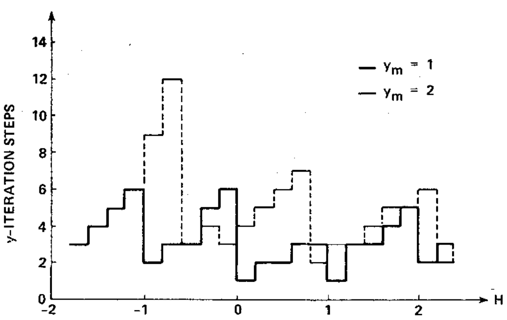

Fig. 4 Number of steps in y-iteration vs H for hyperbolic orbit ($e = 10$).

the error in the extrapolated position and velocity vectors was held to at most one or two digits in the tenth place of decimals. Therefore, it is the efficiency of the method on which we concentrate in the ensuing discussion.

Two elliptic orbits of unit semimajor axis were chosen for illustration having eccentricities of 0.5 and 0.8. The epoch eccentric anomaly E_0 was zero, and E was incremented by steps of 0.2 rad for one complete revolution. The graphs of Figs. 1 and 2 show the number of iterations required for calculating y (or, equivalently, the number of distinct values of the function K) plotted against the eccentric anomaly E. Two values of the maximum permissible y were utilized, $y_m = 1$ and 2. The first corresponds to the heavy lines and the second to the lighter lines in the figures.

Generally, the number of such iteration steps increases monotonically with the anomaly difference until y exceeds y_m, at which point the time T is decremented. When this occurs, the epoch is redefined so that $E - E_0$ is again small and the

Table 2 Required values of N for specified accuracy in computing

$$K = 1 - \sum_{n=1}^{N} k_n (y/10)^n$$

y_m	Number of significant digits in K 6	8	10	12	14	16
0.5	4	5	7	8	10	11
1.0	5	7	9	11	13	15
1.5	6	9	11	14	16	18
2.0	7	10	13	16	19	22

Table 3 Coefficients of economized power series for calculating

$$K = \sum_{n=0}^{6} k_n^* y^n$$

over the interval $(-1, 1)$ with maximum error of $\frac{1}{4} \times 10^{-9}$

n	k_n^*
0	1.00000000001
1	−0.10000000174
2	−0.00357142897
3	−0.00023808136
4	−0.00001919250
5	−0.00000172916
6	−0.00000016292

pattern is approximately repeated. This cyclical behavior is readily apparent from the graphs. Clearly, the smaller the value of y_m, then the greater will be the number of decrements required in T to span a given change in anomaly, and the smaller will be the maximum number of iterations required to converge on the correct value of y.

More specifically, refer to Fig. 1 and consider an eccentric anomaly difference of $E - E_0 = 2.6$. We see that for $y_m = 1$, two decrements in the time are required together with three calculations of y to achieve convergence. On the other hand, for $y_m = 2$, only one decrement in T is needed, followed, however, by five iterations on y.

To obtain the stated accuracy goal in the computation of the final state vector, it was sufficient to terminate the iteration on y when the sequence of approximations ceased to change in the ninth decimal. In contrast, the tolerance on the x-iteration to solve the cubic by Newton's method was found to be comparatively loose, e.g., agreement between successive approximations to only three decimals was sufficient to maintain the accuracy standard. As a sequence, just two or three sequentially calculated values of x were required, on the average, for each iteration cycle in the detemination of y.

In Figs. 3 and 4, similar results for two hyperbolic orbits of unit semimajor axis having eccentricities of 2 and 10 are presented. The epoch value of H_0 was chosen as -2 and H was incremented by steps of 0.2 until $H = 2.4$. The general characteristics of these graphs correspond to the elliptic cases.

With both ellipses and hyperbolas, there is a tendency for the convergence to large values of y to be somewhat slower when the orbital motion is near to and approaching pericenter. This characteristic is manifested quite dramatically in Figs. 1, 3 and 4. No explanation is offered; we simply observe that, in this range, both σ_0 is negative and the motion is changing fairly rapidly.

The orbits chosen for illustration are quite typical. The behavior of the algorithm for circular orbits ($e = 0$) is essentially the same as that depicted in Fig. 1. Needless to say, convergence is instantaneous for parabolic orbits and rapid for near parabolic cases to which the technique was originally applied by Gauss. Rectilinear paths offer no surprises; their convergence properties are indistinguishable from the more familiar orbits. Finally, we note with some satisfaction that the ellipse with semimajor axis $a=9/10$ and eccentricity $e=1/9$, giving rise to a value of $\beta_0=0$, was also unremarkable in its convergence characteristics. (Recall that the possibility of a vanishingly small β_0 motivated the transformation defined by Eq. (20).)

The value to be selected for y_m is intimately related to the number of terms required in the power series for generating the K function. In Table 2, the number of such terms is tabulated for various values of y_m and various accuracy requirements on computing K. Thus, in the examples chosen for illustration in this paper, nine and thirteen terms are seen to be necessary.

If it is desirable to improve further the mechanization efficiency of the general algorithm, the power series for K may be economized using Chebyshev polynomials. The set of such coefficients for an economized series computation of K, valid over the interval $(-1, 1)$ and possessing a maximum error of only $\frac{1}{4}\times 10^{-9}$, is given in Table 3. Thus, a sixth-order polynomial is sufficient to satisfy the accuracy requirements postulated for the examples provided here.

Conclusions

The Kepler algorithm presented in this paper compares favorably with a corresponding subroutine which was prepared for the avionics computers of the NASA Space Shuttle. The latter routine consisted of a fairly standard Newton-Raphson iteration on a form of Kepler's equation which utilized a set of "universal" or "regularized" variables. Both methods were programmed in the same high—order language HAL—the adopted language of the shuttle – and executed with the same precision. Memory utilization was almost exactly the same for the two computer programs. However, in execution time the new method exhibits some definite advantages. For example, if the time difference $t-t_0$ is small enough so that no decrement in T is necessary, the new algorithm uses only 40% to 85% of the time required by the older method. The lowest percentages are typical of the near parabolic and near rectilinear orbits while the largest percentages are characteristic of the hyperbolic cases. If several time increments are required, this computational advantage tends to diminish until, for extreme cases necessitating five or six decrements in T, the percentage can exceed 100%.

The new algorithm was also exercised for a family of ballistic missile trajectories having various ranges from 500 to 7500 nautical miles. Again, the execution time for these cases was found to vary between 65% and 85% of the corresponding time required by the shuttle Kepler algorithm.

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